THE USE OF MATRIX GAMES METHODS, LINEAR PROGRAMMING AND PROBABILITY THEORY IN PLANNING MILITARY OPERATIONS

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The use of matrix game solutions suggested by the author, as well as the methods of linear programming and probability theory in finding solutions to the problem of planning the defeat of military are shown in the article. Mathematical models and optimal strategy of behavior of military conflicts participants are received.

1. The problem of military object defeat

The party A attacks the object, the party B defends it. The party A has got two planes, the party B has got four antiaircraft instruments. Each plane is the carrier of some powerful weapon. Planes of the party A can choose any of four directions to approach the object. The party B can place any of the instruments in any direction, thus each instrument rakes only the area concerning the given direction, and does not rake the other directions. Each instrument can fire only one plane, and the plane under fire is defeated with the probability of 1. The party A does not know, where instruments are placed, and the party B does not know, where the planes will arrive from. The purpose of the party A is to defeat the object, the purpose of the party B is not to admit its defeat. Consequently, the prize is the probability of defeating the object, and we have to find a population median of the prize and optimum strategy of the opponents.

Solution. The conflict between the parties A and B is on. There are only two strategies for the party A: A_1 is to send one plane in two various directions; A_2 is to send both of the planes in one direction.

Possible strategies of the party B are the following five: B_1 is to put one instrument in each direction; B_2 is to put two instruments in two various directions; B_3 is to put two instruments in one direction and one instrument in other two directions; B_4 is to put three instruments in one direction and one in some other direction; B_5 is to put all four instruments in one direction. We shall construct a matrix.

For the definition of probability of event C we shall take advantage of the known formula:

$$P(C_1+C_2)=P(C_1)+P(C_2)-P(C_1C_2), \quad (\Delta)$$

fair for any two casual events. As strategy A_1 of the party A means that planes go in two various directions, the events C_1, C_2 are independent. And therefore the formula can be rewritten as follows

$$P(C_1+C_2)=P(C_1)+P(C_2)-P(C_1)P(C_2). \quad (\Delta\Delta)$$

For calculation of elements of first line h_{1j} of matrix $H_{2\times 5}$ we shall take advantage of this formula. We shall find, for example, element h_{13}. As the party A directs planes in two various directions, and the party B puts two instruments in one direction and one instrument - in other two directions, which makes the probabilities of events C_1, C_2 identical (only one direction from four remains not protected) and are equal $\frac{1}{4}$. Hence,

$$P(C)=h_{13}=\frac{1}{4}+\frac{1}{4}-\frac{1}{4}\times\frac{1}{4}=\frac{1}{2}-\frac{1}{16}=\frac{7}{16}.$$

Let's similarly find other elements h_{1j}:

$$h_{11}=0,\ h_{12}=\frac{3}{4},\ h_{14}=\frac{3}{4},\ h_{15}=\frac{15}{16}.$$

For the calculation of elements h_{2j} of the second line of matrix $H_{2\times 5}$ it is necessary to

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consider that the events C_1, C_2 are dependent. It seems illogical to use the similar formula for dependent events $P(C_1 + C_2) = P(C_1) + P(C_2) - P(C_1)P_{C_1}(C_2)$ or $P(C_1 + C_2) = P(C_1) + P(C_2) - P(C_2)P_{C_2}(C_1)$ in view of uncertainty of the probabilities of events. It is enough to use common sense for probability-theoretic estimation of the considered situations and the formula $P(C) = \frac{k}{n}$ for the calculation of classical probability of casual event C.

As a result we shall receive: $h_{21} = 1$, $h_{22} = \frac{1}{2}$, $h_{23} = \frac{3}{4}$, $h_{24} = \frac{3}{4}$, $h_{25} = \frac{3}{4}$.

So, matrix $H_{2\times5}$ looks like:

$$H_{2\times5} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{7}{16} & \frac{3}{4} & \frac{15}{16} \\ 1 & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \end{pmatrix}.$$

It is obvious, that the fourth vector-column surpasses the third vector-column, therefore it is possible to delete the fourth column from the matrix. Similarly, the fifth vector-column surpasses the third vector-column. Hence, it is possible to remove the fifth column as well.

$x = (x_1, x_2)$ $z = t_1 + t_2 \to min$

So, the game with matrix $H_{2\times5}$ can be shown as a game with the matrix

$$H_{2\times3} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{7}{16} \\ 1 & \frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

As the bottom price of this game $v_1 = \frac{1}{2}$ does not coincide with the top by $v_2 = 1$, for finding the solution to the game the mixed strategy is used. To find the optimum mixed strategy of the first player and the price of the game v, we shall make a problem of linear programming. Let be any mixed strategy of the *I* player (party *A*). Numbers x_1, x_2 satisfy the following restrictions:

$$\begin{cases} x_2 \ge v, \\ \frac{3}{4}x_1 + \frac{1}{2}x_2 \ge v, \\ \frac{7}{16}x_1 + \frac{3}{4}x_2 \ge v, \\ x_1 + x_2 = 1, \\ x_1 \ge 0, x_2 \ge 0. \end{cases}$$

Having divided all these parities by number $v > 0$ and after entering the designations $t_i = \frac{x_i}{v}$, $i = 1,2$, we shall receive the following problem of linear programming

$$\begin{cases} t_2 \ge 1, \\ \frac{3}{4}t_1 + \frac{1}{2}t_2 \ge 1, \\ \frac{7}{16}t_1 + \frac{3}{4}t_2 \ge 1, \\ t_1 \ge 0, t_2 \ge 0. \end{cases} \qquad (1)$$

Problem 1 can be solved with simplex-method. Then we shall transform the system of the equations to the basic kind, applying simplex transformations to choose resolving elements.

Let's express a target linear function through free unknown persons t_3, t_5 of the previous system, the result will be this function in the form of $z = \frac{47}{21} - \frac{1}{21}t_3 + \frac{16}{7}t_5$.

Then we shall make a simplex-table1.

Analyzing a simplex-table and applying an attribute of an optimality of the basic plan of a problem of linear programming we come to the conclusion that the received basic plan $t^1_{onop} = \left(\frac{26}{21}, 1, 0, \frac{3}{7}, 0\right)$ is not optimum. After that we choose the resolving element in the column of the table corresponding to unknown person t_3, on a way of simplex transformations, having counted elements of the table by the rule of rectangle, we shall receive the second a simplex-table 2.

The received basic plan $t^2_{onop} = \left(0, \frac{24}{11}, \frac{13}{11}, \frac{1}{11}, 0\right)$ (on the basis of op-

Table 1

Basic unknown	Free members	t_1	t_2	t_3	t_4	t_5
t_1	$\frac{26}{21}$	1	0	$\frac{22}{21}$	0	$-\frac{16}{7}$
t_2	1	0	1	-1	0	0
t_4	$\frac{3}{7}$	0	0	$\frac{2}{7}$	1	$-\frac{12}{7}$
z	$\frac{47}{21}$	0	0	$\frac{1}{21}$	0	$-\frac{16}{7}$

Table 2

Basic unknown	Free members	t_1	t_2	t_3	t_4	t_5
t_3	$\frac{13}{11}$	$\frac{21}{22}$	0	1	0	$-\frac{24}{11}$
t_2	$\frac{24}{11}$	$\frac{21}{22}$	1	0	0	$-\frac{24}{11}$
t_4	$\frac{1}{11}$	$-\frac{3}{11}$	0	0	1	$-\frac{12}{11}$
z	$\frac{24}{11}$	$-\frac{1}{22}$	0	0	0	$-\frac{24}{11}$

timality) is optimum, and also a unique optimum plan as the factors corresponding to basic unknown persons t_2, t_3, t_4 are equal in the last line of a simplex-table to zero only.

Considering that $t_1 + t_2 = \frac{1}{v}$, and $t_1 = 0$, $t_2 = \frac{24}{11}$, we shall calculate the price of game v: $v = \frac{11}{24}$.

Thus, after our research there was an opportunity to give out the most suitable recommendation to the party *A*, that is aspiring to defeat a military object rationally. To reach it the party *A* should use the second strategy A_2, i.e. send both of the planes in one direction from four directions with probability *1*. In that case the party *A* will achieve the guaranteed average prize: the population mean of defeat of the military object will make $\frac{11}{24}$.

It is necessary to find optimum strategy $y^* = (y_1^*, y_2^*, y_3^*)$ for the second player in the game with the matrix

$$H_{2\times 3} = \begin{pmatrix} 0 & \frac{3}{4} & \frac{7}{16} \\ 1 & \frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

The problem of linear programming for a finding the strategy y^* looks like:

$$w = u_1 + u_2 + u_3 = \frac{1}{v} \to max,$$

$$\begin{cases} \frac{3}{4}u_2 + \frac{7}{16}u_3 \le 1, \\ u_1 + \frac{1}{2}u_2 + \frac{3}{4}u_3 \le 1, \\ u_1 \ge 0, u_2 \ge 0, u_3 \ge 0, \end{cases}$$

where $u_j = \frac{y_j}{v}$, $j = 1,2,3$, is the price of game, is the mixed strategy of the second player.

For a finding of optimum strategy y^* of the second player we shall take advantage of the second theorem of a duality. For this purpose the preliminary restrictions - the inequalities of problems I and II we shall write down in the form of equations, resorting to the introduction of additional unknown persons. We shall have as a result the following two systems of equations:

$$\begin{cases} t_2 - t_3 = 1, \\ \frac{3}{4}t_1 + \frac{1}{2}t_2 - t_4 = 1, \\ \frac{7}{16}t_1 + \frac{3}{4}t_2 - t_5 = 1, \end{cases} \quad (1)$$

$$\begin{cases} \frac{3}{4}u_2 + \frac{7}{16}u_3 + u_4 = 1, \\ u_1 + \frac{1}{2}u_2 + \frac{3}{4}u_3 + u_5 = 1. \end{cases} \quad (2)$$

Let's take advantage of an optimum plan $t^*_{onm} = \left(0, \frac{24}{11}, \frac{13}{11}, \frac{1}{11}, 0\right)$ of the transformed problem I found above for finding the optimum strategy y^*. As coordinates $t^*_2 = \frac{24}{11}$, $t^*_3 = \frac{13}{11}$, $t^*_4 = \frac{1}{11}$ are positive numbers, under the second theorem of duality the corresponding coordinates of optimum solution to $u^* = \left(u^*_1, u^*_2, u^*_3, u^*_4, u^*_5\right)$ of the transformed problem II are equal to zero, i.e. $u^*_1 = 0$, $u^*_2 = 0$, $u^*_5 = 0$. The coordinate u^*_3 we shall find from the equality $u_1 + u_2 + u_3 = \frac{1}{v}$ in which we shall substitute the numbers found $u^*_1 = 0$, $u^*_2 = 0$, $v = \frac{11}{24}$. We shall receive $u^*_3 = \frac{24}{11}$. Having substituted the numbers of $y_1 = u_1 v$, $y_2 = u_2 v$, $y_3 = u_3 v$ in the equality, we shall receive the optimum strategy $y^* = (0,0,1)$ of the player II in the game with the matrix $H_{2\times 3}$. It is obvious, that the optimum strategy of the second player in the game with the initial matrix $H_{2\times 5}$ following: $y^* = (0,0,1,0,0)$. Thus, the party *B*, defending a military object, should apply the third strategy B_3 with the probability equal to *1*, i.e. the best way to reflect an attack of the opponent for the party *B* is to put two instruments in one direction and one instrument in other two directions with the probability of *1*.

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