CONFLICT MODELLING BY RECTANGULAR PLAY AND THE APPLICATION OF THE SOLUTIONS TO THE APPLIED PROBLEMS OF ECONOMY AND MILITARY SCIENCE

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The conditions for modeling a real-life antagonistic conflict by a finite antagonistic play are studied. The problem of antagonistic competition is solved, thus illustrating the principle of modeling the conflict between two firms by means of a rectangular game and allowing to find the optimum mixed strategy of this game by the means of methods of linear programming, probability theory, the properties of the solutions to the rectangular game received by the author of this article, and to simplify the rectangular game. Various ways of applying the theory of rectangular games to applied problems of economy and military science are shown.

Any real conflict can be modeled by finite antagonistic play if it satisfies the following conditions:

1) the conflict is defined by antagonistic interaction of two parties, each having only finite amount of possible actions;

2) the actions are taken independently, i.e. each of the parties has no information on the action made by the other party; the result of these actions is estimated by material number which defines the utility of the situation for one of the parties, thus the utility of such situation for the other party equals this number with a negative sign;

3) each of the parties estimates the utility of any possible situation which can develop as a result of their interaction both for itself, and for the opponent;

4) the structure of the actions of the parties in conflict has no formal distinctive properties, it allows to interpret the actions of the parties as the elements of some abstract sets, distinguishing various actions from each other only to a degree of utility of the developed situation.

If the conflict satisfies the four listed conditions, it is possible to describe this conflict as finite antagonistic play $\Gamma = <X, Y, H>$, where X is the set of pure strategy of the 1st player, Y is the set of pure strategy of the 2nd player, H is the payoff functional (utility) of the 1st player, certain for all the pairs (x, y) of possible pure strategy of the players ($x \in X$, $y \in Y$). Conditions 1-4 allow presenting the conflict in the form of a game. The finite antagonistic play $\Gamma = <X, Y, H(x, y)>$ can be set in the form of a matrix $H_{m\times n} = (h_{ij})$, where m is the number of pure strategy of the 1st player $(i = 1,2,\dots, m)$, n is the number of pure strategy of the 2nd player $(j = 1,2,\dots, n)$, $h_{ij}=H(i,j)$ is the prize of the 1st player in a situation (i,j). Matrix $H_{m\times n} = (h_{ij})$ is a game-theory model (rectangular game) of real conflicts which satisfy the conditions 1-4. Its construction and the subsequent analysis can enrich the theory of finite antagonistic plays in economy, military science, sports, biology, etc.

In the book «The Theory of games and economic behavior» (Princeton, 1944) a mathematician John von Neumann and an economist Oscar Morgenstern described the grounds of the mathematical theory of «the conflict of interests». Based on the assumption that human actions in the economic sphere are purposeful actions, they started off with the introduction of the concept of «utility», necessary for the measurement of the results of actions. The set of axioms by D. Neumann and O. Morgenstern uniquely defines the habitual word «utility».

Let's consider a model problem of antagonistic competition.

Problem: the firm A in the conditions of capitalist economy makes some seasonal goods (for example, female boots) which has demand during n time units, for example, 6 months, and acts on the market during moment $i(i = 1,2,3,4,5,6)$. The second firm B, produces the same goods and, having the purpose to ruin firm A, prepares for the release and sale of the goods during the chosen period of time. It is

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supposed, that quality of the competing goods depends on the time - the later the goods enter the market, the higher their quality is, and only the goods of higher quality are sold well, and the income of sale per unit of time makes k dollars.

We must calculate the expected value of the income of firm A so that the income of firm A maximizes due to successful sales per specified time units, pursuing the purpose of minimizing the income of firm B.

Solution: it is obvious that the firms A, B, having the opposite purposes, are in a situation of conflict. We shall construct a game-theory model of this conflict, while their sets of strategy coincide $X = Y = \{1,2,3,\dots,n\}$. We shall calculate the payoff functional $H(i,j)$ for the 1st player. Let's assume, that the firm A will launch the goods during the i moment of time, and firm B - during the j moment of time, and $j > i$. Then the firm A, having no competitor during time units, will receive the income of dollars for this period of time. If during the j moment of time the firm B launches the goods of higher quality, then the firm A loses the market and does not receive $h_{43} = k(6-4+1) = 3k,$ any further income. However, if the firm A, having caught the proper moment $i > j$ launches still better goods, it will receive the income during the remaining time units on the market. This income will equal $k(n-i+1)$ dollars.

So, the payoff functional of the 1st player is set as follows:

$$H(i,j) = \begin{cases} k(j-i), & i < j; \\ \frac{1}{2}k(n-i+1), \text{ if } & i = j; \\ k(n-i+1), & i > j. \end{cases}$$

Let's write down the matrix H of the received finite antagonistic play for the general case:

$$H_{n\times n} = \begin{pmatrix} \frac{kn}{2} & 1k & 2k & \dots & k(n-2) & k(n-1) \\ k(n-1) & \frac{k(n-1)}{2} & 1k & \dots & k(n-3) & k(n-2) \\ k(n-2) & k(n-2) & \frac{k(n-2)}{2} & \dots & k(n-4) & k(n-3) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2k & 2k & 2k & \dots & 1k & 1k \\ 1k & 1k & 1k & \dots & 1k & \frac{1}{2}k \end{pmatrix}.$$

We shall not solve this game in general, and we shall be limited to the solution of such rectangular game for a case, when $n = 6$. Matrix H will look like:

$$H_{6\times 6} = \begin{pmatrix} 3k & 1k & 2k & 3k & 4k & 5k \\ 5k & \frac{5}{2}k & 1k & 2k & 3k & 4k \\ 4k & 4k & 2k & 1k & 2k & 3k \\ 3k & 3k & 3k & \frac{3}{2}k & 1k & 2k \\ 2k & 2k & 2k & 2k & 1k & 1k \\ 1k & 1k & 1k & 1k & 1k & \frac{1}{2}k \end{pmatrix}.$$

Let's track how the 4th line of matrix $H_{6\times 6}$ fills out, i.e. how its elements $h_{41}, h_{42}, h_{43}, h_{44}, h_{45}, h_{46}$ appear. For this purpose we consistently address the payoff functional , considering the three cases $(i < j, i = j, i > j)$:

$h_{41} = k(6-4+1) = 3k,$ $h_{42} = k(6-4+1) = 3k,$

$h_{44} = k\frac{1}{2}(6-4+1) = k\frac{3}{2},$ $h_{45} = k(5-4) = k1,$ $h_{46} = k(6-4) = 2k$. If the all elements of the matrix $H_{6\times 6}$ are divided by the number $k \neq 0$ we shall receive the matrix

$$H^1_{6\times 6} = \begin{pmatrix} 3 & 1 & 2 & 3 & 4 & 5 \\ 5 & \frac{5}{2} & 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 1 & 2 & 3 \\ 3 & 3 & 3 & \frac{3}{2} & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \frac{1}{2} \end{pmatrix},$$

which sets the game, whose optimum mixed strategy coincide with the optimum strategy of the game with the matrix $H_{6\times 6}$, but the value of the game with the matrix $H^1_{6\times 6}$ decreases in k time in comparison with the game with the matrix $H_{6\times 6}$. Now we shall consistently simplify the rectangular game with the matrix $H^1_{6\times 6}$.

Let $x^* = \left(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*\right)$ be the optimum mixed strategy of the 1st player, and $y^* = \left(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*, y_6^*\right)$ be the optimum mixed strategy of the 2nd player, $v^*_{H^1_{6\times 6}} = H\left(x^*, y^*\right)$ be the price of the game with the matrix $H^1_{6\times 6}$. Apparently, the elements of the 6th line of the matrix $H^1_{6\times 6}$ are less than the corresponding elements of the 5th line, therefore the 5th strategy of the 1st player dominates the 6th strategy, elements of the 1st column of matrix $H^1_{6\times 6}$ not less corresponding elements of the 2nd column, therefore the 2nd strategy of the 2nd player dominates the 1st strategy. After the deletion of the 1st column and the 6th line we shall receive a matrix

$$H^2_{5\times 5} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{5}{2} & 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 & 3 \\ 3 & 3 & \frac{3}{2} & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

As the convex linear combination of the 1st and the 3rd vectors-lines with factors $\frac{1}{2}$ and surpasses the 2nd vector-line the 2nd line can be rejected. Besides, the elements of the 5th column are not less than the corresponding elements of the 4th column, therefore it is possible to remove the 5th column. As a result we shall receive the matrix

$$H^3_{4\times 4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 \\ 3 & 3 & \frac{3}{2} & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}.$$

The fourth line of the matrix $H^3_{4\times 4}$ can be removed, as a convex linear of combination the 1st, the 2nd and 3rd vector-lines with the factors $\frac{1}{2}$, 0, $\frac{1}{2}$ surpasses the 4th vector-line. Analyzing the matrices $H^1_{6\times 6}$ and $H^3_{4\times 4}$ using the properties of the solutions to the corresponding games, we conclude, that with the optimum mixed strategy $x^* = \left(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*\right)$ of the 1st player the 2nd, the 5th and the 6th coordinates are equal to zero: $x_2^* = x_5^* = x_6^* = 0$. To find other coordinates, we shall find the optimum mixed strategy of the 1st player with the matrix $H^4_{3\times 4}$ received from the matrix $H^3_{4\times 4}$ by the deletion of the 4th line:

$$H^4_{3\times 4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 2 \\ 3 & 3 & \frac{3}{2} & 1 \end{pmatrix}.$$

It is obvious, that the game with the matrix $H^4_{3\times 4}$ has no solution in pure strategy as the bottom price of the game is not equal to the top price of the game.

Let's reduce the game with the matrix $H^4_{3\times 4}$ to a problem of linear programming. The coordinates of optimum strategy of the 1st player $\tilde{x}^* = \left(x_1^*, x_3^*, x_4^*\right)$ satisfy the following restrictions:

$$x_1^* + 4x_3^* + 3x_4^* \ge v,$$

$$2x_1^* + 2x_3^* + 3x_4^* \ge v,$$

$$3x_1^* + x_3^* + \frac{3}{2}x_4^* \ge v,$$

$$4x_1^* + 2x_3^* + x_4^* \ge v,$$

$$x_1^* \ge 0,\ x_3^* \ge 0,\ x_4^* \ge 0,$$

$$x_1^* + x_3^* + x_4^* = 1,$$

where v is the price of the game, and ,

as all elements of matrix $H^4_{3\times 4}$ are not negative. Having divided all the members of the restrictions by number v, by entering the designation , $i = 1,3,4$, we shall receive the following problem of linear programming:

$$z = t_1 + t_2 + t_3 = \frac{1}{v} \to \min$$

$$\begin{cases} t_1 + 4t_2 + 3t_3 \ge 1, \\ 2t_1 + 2t_2 + 3t_3 \ge 1, \\ 3t_1 + t_2 + \frac{3}{2}t_3 \ge 1, \\ 4t_1 + 2t_2 + t_3 \ge 1, \end{cases}$$

$$t_1 \ge 0, \ \ t_2 \ge 0, \ \ t_3 \ge 0$$

Let's solve problem (1) via simplex-method. For this purpose the problem will be reduced to the initial form:

$$z = t_1 + t_2 + t_3 \to \min$$

$$t_1 + 4t_2 + 3t_3 - t_4 = 1,$$

$$2t_1 + 2t_2 + 3t_3 - t_5 = 1,$$

$$3t_1 + t_2 + \frac{3}{2}t_3 - t_6 = 1,$$

$$4t_1 + 2t_2 + t_3 - t_7 = 1,$$

$$t_i \ge 0, \ \ i = 1,2,3,4,5,6.$$

By applying simplex transformations to the system of the linear equations (2), we shall lead the system to the basic kind. As a result we receive:

$$t_2 - \frac{1}{2}t_4 + \frac{5}{8}t_5 - \frac{1}{4}t_6 = \frac{1}{8},$$

$$t_3 + \frac{1}{3}t_4 - \frac{11}{12}t_5 + \frac{1}{2}t_6 = \frac{1}{12},$$

$$-\frac{2}{3}t_4 + \frac{4}{3}t_5 - 2t_6 + t_7 = \frac{1}{3},$$

$$t_1 + \frac{1}{4}t_5 - \frac{1}{2}t_6 = \frac{1}{4}.$$

Expressing the basic unknown variables t_1, t_2, t_3 through free unknown variables t_4, t_5, t_6 and substituting the received expressions in target linear function $z = t_1 + t_2 + t_3$, we shall receive this function in the form of

$$z = \frac{11}{24} + \frac{1}{6}t_4 + \frac{1}{24}t_5 + \frac{1}{4}t_6.$$

Further on, we shall make a simplex-table for finding optimum plans of problem (2):

$\frac{11}{24}$ $t_i = \frac{x_i^*}{v}$

basic variables	Free variables	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	$-\frac{1}{2}$	0
t_2	$\frac{1}{8}$	0	1	0	$-\frac{1}{2}$	$\frac{5}{8}$	$-\frac{1}{4}$	0
t_3	$\frac{1}{12}$	0	0	1	$\frac{1}{3}$	$-\frac{11}{12}$	$\frac{1}{2}$	0
t_7	$\frac{1}{3}$	0	0	0	$-\frac{2}{3}$	$\frac{4}{3}$	-2	1
z	$\frac{11}{24}$	0	0	0	$-\frac{1}{6}$	$-\frac{1}{24}$	$-\frac{1}{4}$	0

Analyzing the simplex-table and applying an attribute of optimality of the basic plan, we conclude, that the received basic plan $T = \left(\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, 0,0,0, \frac{1}{3}\right)$ is the unique optimum plan of the problem of linear programming (2), and the value of criterion function on this plan equals . Considering that $z = t_1 + t_2 + t_3 = \frac{1}{v}$, and $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{8}$, $t_3 = \frac{1}{12}$, we shall receive $v = \frac{24}{11}$. Now, knowing that $t_i = \frac{x_i^*}{v}$, we shall find $x_1^* = t_1 \times v = \frac{6}{11}$, $x_3^* = t_2 \times v = \frac{3}{11}$, $x_4^* = t_3 \times v = \frac{2}{11}$. So, the optimum mixed strategy $\tilde{x}^*$ of the first player in the game with the matrix $H^4_{3\times 4}$ is found: $\tilde{x}^* = \left(\frac{6}{11}, \frac{3}{11}, \frac{2}{11}\right)$. Hence, $x^* = \left(\frac{6}{11}, 0, \frac{3}{11}, \frac{2}{11}, 0, 0\right)$ is the optimum mixed strategy of the first player (firm A) of the initial rectangular game, and the income of the firm from selling the seasonal goods will make $vk = \frac{24}{11}k$ dollars.

It is necessary to make recommendations for the firm A. Not to be ruined by the firm B, it is necessary to deliver the seasonal goods on the market in the 1st, 3rd and the 4th time unit according to the probabilities equal to $\frac{6}{11}, \frac{3}{11}, \frac{2}{11}$.

Finding of the optimum mixed strategy of the second firm can be carried out similarly.

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