ATTRIBUTES OF DECISIONS OF THE MATRIXAL GAME AND THEIR USE FOR MAKING DECISIONS IN CONDITIONS OF INDEFINITY

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Attributes of solutions of the matrixal game are examined, developed and formed by the author in eight theorems. These attributes are used to simplify the matrixal game, to solve this game by developing and making decisions in conditions of indefinity, in conflict situations.

Let's study the problem of decision making in economic or any other system[1]. Three objects are usually distinguished: 1) the object of management (a managed subsystem), 2) a managing subsystem, 3) environment. A managing subsystem influences the object of management by a variety of managing impacts. The method of solving the problems of decision making on the base of mathematical modeling consists of three stages: 1) constructing mathematical model of the problem of decision-making; 2) finding the optimal decision in accordance with the chosen optimal principal; 3) the analysis of the results. In order to construct such a model it is necessary to examine three multitudes: 1) the multitude of X alternatives; 2) the multitude of Y re-

$x_i \in X$, $\langle X, Z, F \rangle$, $Y = \{y_{11}, y_{12}, \dots, y_{mn}\}$

sults; 3) the multitude of Z possible conditions of the environment. Any result $y \in Y$ considerably depends on the choice of the alternative $x \in X$ and the possible state of the environment $z \in Z$. That is why in common case we will consider that there is function $y = F(x,z)$, characterizing this dependency, and it is assumed that parameter Z is unknown in the moment of decision making. Function $y = F(x,z)$ is called the function of implementation or reflection $F : X \times Z \to Y$, that for every ordered couple of type (x,z) suggests the corresponding result y. The set of objects makes the implementation structure of the problem of decision making, showing the relation between the chosen alternatives, the condition of environment, settings and the got results. Let's consider the case, when several managing subsystems and opponents having controversial interests behave actively in the conditions of uncertainty. In this case the problem of decision making is described by the mentioned function of implementation $y = F(x,z)$ in the situation, when one subject chooses the alternative $x \in X$, and another subject chooses the alternative $z \in Z$ in the conditions of conflict and in the conditions of uncertainty: the choice of the alternative by one subject characterizes the uncertainty of the environment for another subject, i.e. the subjects are the conditions of uncertainty of the type "active partner". The problem of decision making in the conditions of such uncertainty is accomplished by game theory - mathematical discipline, whose subject is the mathematical models of conflict situations. If the multitude of alternatives X, multitudes of results Y and multitude Z of the conditions of the environment are finite, the situation of the choice of the alternative in the conditions of uncertainty can be represented with the help of the table (or matrix A), illustrating the behavior of function $y = F(x,z)$, where $x \in X$, $z \in Z$, $y \in Y$. Let's assume, that multitude X, Z are the folowing: $X = \{x_1, x_2, \dots, x_m\}$, $Z = \{z_1, z_2, \dots, z_n\}$. Then multitude Y has mn elements: .
The multitude of results Y interpreting matrix $Y_{m \times n}$ is of size $m \times n$. If the alternative is chosen, and the state of the environment in such situation is Z_j, then the corresponding result Y_{ij} is in matrix on the crossing of i-line and j-column. Let's pay attention that many problems of game theory are solved in the frames of matrix games that are set by their matrixes. In this article the author reveals the results of the research targeted at revealing the solutions of a matrix game.

The solution of matrix game is the number of objects $\left\{x^*, y^*, V_* = H\left(x^*, y^*\right)\right\}$,

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where x^* - the optimal mixed strategy of the first player, y^* - optimal mixed strategy of the second player, $V_* = H\left(x^*, y^*\right)$ - the prize of the first player in situation $\left(x^*, y^*\right)$, called as a value of matrix game. In game $\Gamma =< X, Y, H(x, y) >$ any strategies x^*, y^* $\left(x^* \in X, y^* \in Y\right)$ are called optimal , if the inequality is obeyed

$$H\left(x, y^*\right) \le H\left(x^*, y^*\right) \le H\left(x^*, y\right),\ x \in X,$$
$$y \in Y.$$

Matrix game is simplified due to the possible decrease of the size of its matrix, if lines and columns of this matrix are connected by certain correlation.

The author of the article formulates and proves eight theorems that show the characteristics of a set of objects $\left\{x^*, y^*, V_*\right\}$, being the solution of a matrix game.

Theorem 1. In order to make the set of objects $\left\{x^*, y^*, V_*\right\}$ be part of game solution in mixed strategies with matrix $A_{m\times n}$, it is necessary and enough to fulfill the inequality

$$\left(e^i_{m\times 1}\right)' A_{m\times n} y^*_{n\times 1} \le V_* \le \left(x^*_{m\times 1}\right)' A_{m\times n} e^j_{n\times 1},$$
$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n,$$

where $e^i_{m\times 1}$ - i- single ort of m-measured Evklid space ; $e^j_{n\times 1}$ - j- single ort of n-measured Evklid space ; "**r**" — the operation of matrix transposition; $y^*_{n\times 1}$ - matrix, having n lines and one column (the optimal mixed strategy of the second player); - matrix having m lines and one column (the optimal mixed strategy of the first player); - the price of the game; $A_{m\times n}$ - matrix having m lines and columns.

Theorem 2. If the set of objects is the solution of the game with prize matrix $A_{m\times n}$ and if k - coordinate of the optimal mixed strategy x^* of the first player is positive $\left(x^*_k > 0\right)$, and $a_{k1}y^*_1 + a_{k2}y^*_2 + \ldots + a_{kn}y^*_n = V_*$, $y^* = \left\{y^*_1, y^*_2, \ldots, y^*_n\right\}$, then the following equality is fulfilled:

$$V_* = \left(e^k_{m\times 1}\right)' A_{m\times n} y^*_{n\times 1},$$

where V_* - the price of the game, $y^*_{n\times 1} = \left\{y^*_1, y^*_2, \ldots, y^*_n\right\}$ - optimal mixed strategy of the second player, $e^k_{m\times 1}$ - k-single ort of m-measured Evklid space .

Theorem 3. If the multitude of objects $\left\{x^*, y^*, V_*\right\}$ is the solution of the game with prize matrix $A_{m\times n}$ and if l-coordinate of y^*_l optimal mixed strategy y^* of the second player is positive $\left(y^*_l > 0\right)$, and $a_{1l}x^*_1 + a_{2l}x^*_2 + \ldots + a_{ml}x^*_m = V_*$, then the following equality is fulfilled:

$$V_* = \left(x^*_{m\times 1}\right)' A_{m\times n} e^l_{n\times 1},$$

where V_* - the price of the game, $X^* = \left\{X^*_1, X^*_2, \ldots, X^*_m\right\}$ - optimal mixed strategy of the first player, e^l - l-single ort of n-measured Evklid space .

Theorem 4. If multitude $\Gamma = \left\{x^*, y^*, V_*\right\}$ - the solution of the game with matrix $A_{m\times n}$ and if k-coordinate of x^*_k vector $x^* = \left\{x^*_1, x^*_2, \ldots, x^*_m\right\}$ equals 0 $\left(x^*_k = 0\right)$, and l- coordinate y^*_l of vector $y^* = \left\{y^*_1, y^*_2, \ldots, y^*_n\right\}$ also equals 0 $\left(y^*_l = 0\right)$, then multitude $\Gamma_1 = \left\{\tilde{x}^*, \tilde{y}^*, V_*\right\}$ is the solution of the game with matrix B, got from matrix $A_{m\times n}$ by deleting k-line and l-column,

where $\tilde{x}^* = \left(x^*_1, x^*_2, \ldots, x^*_{k-1}, x^*_{k+1}, \ldots, x^*_m\right)$,

$\tilde{y}^* = \left(y^*_1, y^*_2, \ldots, y^*_{l-1}, y^*_{l+1}, \ldots, y^*_n\right)$,

$k = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n$.

Definition. Simple solution of the game with matrix $A_{m\times n}$ is such solution $\left\{x^*, y^*, V_*\right\}$, that satisfies two equations

$$V_* = \left(e^i_{m\times 1}\right)' A_{m\times n} y^*_{n\times 1},\ i = 1, 2, \ldots, m;$$

$$V_* = \left(x^*_{m\times 1}\right)' A_{m\times n} e^j_{n\times 1}, \; j = 1,2,\ldots,n.$$

Theorem 5. If game with a squared non-degenerate matrix allows simple solution, then this solution is the only one and looks like this

$$x^* = V_*\left(A'_{m\times n}\right)^{-1} d'_{n\times 1}, \qquad y^* = V_* A^{-1}_{n\times n} d'_{n\times 1},$$

$$V_* = \frac{1}{d_{1\times n} A^{-1}_{n\times n} d_{n\times 1}},$$

where $d_{1\times n} = (1,1,\ldots,1)$.

Theorem 6. If a set of objects $\Gamma_A = \left\{x^*, y^*, V_*\right\}$ is the solution of the game with matrix $A_{m\times n} = (a_{ij})$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ in mixed strategies, then the multitude $\Gamma_B = \left\{x^*, y^*, kV_* + \alpha\right\}$ is the solution of the game in mixed strategies with matrix $B_{m\times n} = (ka_{ij} + \alpha)$, where a number $k > 0$, α - any real number, ; $j = 1,2,\ldots,n$.

Theorem 7. If multitude $\Gamma_B = \left\{x^*, y^*, kV_* + \alpha\right\}$ is the solution of the game with matrix $B_{m\times n} = (ka_{ij} + \alpha)$ in mixed strategies, where number $k > 0$, α - any real number, ; $j = 1,2,\ldots,n$, then multitude $\Gamma_A = \left\{x^*, y^*, V_*\right\}$ is the solution of the game in mixed strategies with matrix $A_{m\times n} = (a_{ij})$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

The correctness of opposite theorems 6 and 7 prove that the following theorem is right that shows the necessary feature, when three objects $\left\{x^*, y^*, V_*\right\}$ is the solution of the game in mixed strategies with matrix $A_{m\times n}$.

Theorem 8. So as the multitude $\Gamma_A = \left\{x^*, y^*, V_*\right\}$ become a solution in mixed strategies with matrix $A_{m\times n} = (a_{ij})$, it is necessary and enough for multitude $\Gamma_B = \left\{x^*, y^*, kV_* + \alpha\right\}$ to become the solution of the game in mixed strategies with matrix $B_{m\times n} = (ka_{ij} + \alpha)$, where k - positive real number $(k > 0)$, α - any real number, ; $j = 1,2,\ldots,n$, x^* - optimal mixed strategy of the first player, y^* - optimal mixed strategy of the second player, V_* - the price of the game with matrix $A_{m\times n}$.

The analysis of the theorems 1 and 8 shows that they reflect the necessary conditions of the solutions for the game $\left\{x^*, y^*, V_*\right\}$ in mixed strategies with matrix $A_{m\times n}$ in different ways.

The theorems 1-8 are widely used for simplifying the matrix game and finding solutions. The thing is that any matrix game set by its matrix $A_{m\times n}$, can be solved in clear and mixed strategies. However finding solutions for matrix game depends on the size of matrix $A_{m\times n}$. That is why we need mathematical methods that will allow to solve one matrix game by the solution of other matrix game, set by matrix B, having smaller size, and therefore the multitude of solutions of the game with matrix B was at least included in the multitude of solutions of the game with matrix $A_{m\times n}$ or coincide with the last multitude.

Comments. In case when the problem of decision making is described by the implementation function $y = F(x, z)$, and "nature", environment manage parameter z, then conflict situations are solved by games with "nature".

If every result should be evaluated by real number, then the composition of two reflections , where $F : X \times Z \to Y$, $J : Y \to R$, R - the multitude of real numbers. Then result y and its evaluation are identified, implementation function is transferred in the material function $J(x, z)$, that is maximized or minimized to x depending on the meaning of the problem being solved and in fact is the functional of two variables. It means that the problem of making decisions can be formulated as the problem of optimization and is solved by the method of optimization.

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[1] See.: *Neyman G., Morgenstern O.* Game theory and economic behavior. M., 1970; *Dubin G.N., Suzdal V.G.* Introduction into applied game theory. M., 1981; *Vorobiev N.N.* Game theory for economist-cyberneticists. M., 1985; *Chegodaev A.I.* The basic features of the theory of finite antagonistic games and their application to solving economic and military problems: Study book. Yaroslavl, 1993; *Kuzmina N.M.* Systematic approach to change managing // Vestnik of Samara State University of Economics, 2007. № 3 (29).