MATHEMATICALSUPPORTFORMAKINGDECISIONSINCONDITIONS OFINDEFINITYBYMEANSOFTIESBETWEENTHEATTRIBUTES OFDECISIONSOFMATRIXALGAMESANDTHEPRINCIPLE OFDOMINATION

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The ties between solutions of two matrixal games if the lines and the columns of matrixes of these games the demands of domination are satisfied are analysed. There are five theorems developed by the authors. The author sets an example how to find a solution of a matrixal game using attributes of decisions of the matrixal games, with ties between decisions and successive simplification of this game.

Mathematical support of decision making in conditions of indefinity can be done by the domination approach, matrix game theory, correlation between the solutions of two matrix games, determined by their matrix, lines or columns of which are satisfying the demand of domination[1].

The principle of maximin and the principle of equilibrium are usually used while searching the solution of the matrix game. However these principals applied for a mixed expansion of the matrix game are equal. We will be automatically consistent with the principle of domination, which is more natural and easy, if we follow the principle of maximin and the principle of equilibrium. The principal of domination: player should not use with the positive probability clear strategies, in case of using which he is winning less than by using any other strategy. In contrast to the principle of maximin and the principle of equilibrium, the domination principle does not lead to the understanding of the situation which is relevant to this principle. The domination principle is a principle of the prohibition.

X, Y $x^B = \left(x_1^B, x_2^B, ..., x_{i-1}^B, x_i^B, ..., x_{m-1}^B\right)$

We will consider three definitions which are relevant to the domination principle.

Definition 1. It is said that an arithmetic vector $a = (a_1, a_2, ..., a_n)$ dominates an arithmetic vector $b = (b_1, b_2, ..., b_n)$, if $a_i \ge b_i \, (i = 1, 2, ..., n)$; if $a_i > b_i$ for every i, than strongly dominates .

Definition 2. The strategy x^1 of the first player strictly dominates the strategy x^2, if inequation is valid, i.e. $H(x^1, y) > H(x^2, y)$, $\forall y \in Y$, $x^1, x^2 \in X$, H - function of the win; y - any strategy of the second player; - a host of strategies of the first and second players accordingly.

Definition 3. The strategy y^1 of the first player strictly dominates the strategy y^2, if $H(x, y^1) < H(x, y^2)$, $\forall x \in X$; $y^1, y^2 \in Y$.

There are five theorems developed and proved by the author of an article. These theorems establish the ties between solutions of two matrix games if the lines and the columns of matrixes of these games satisfy the demand of domination and some other conditions.

Theorem 1. Let i-line of a matrix is dominated by a convex linear combination of other lines of this matrix and let B is a matrix received from a matrix $A_{m \times n}$ by removal of i-line, set $\{x^B, y^B, v_B\}$ is the decision of the game with a matrix B, V_B is the game price, x^B is the optimum mixed strategy of the first player, y^B is the optimum mixed strategy of the second player, and $x^* = \left(x_1^B, x_2^B, ..., x_{i-1}^B, 0, x_i^B, ..., x_{m-1}^B\right)$ is the arithmetic vector of dimension m received from a vector of dimension $m-1$ by addition a coordinate equal to zero and located between coordinates x_{i-1}^B and x_i^B with numbers $i-1$ and i.

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Then the set $\{x^*, y^B, v_B\}$ is the decision of the game with a matrix $A_{m\times n}$.

Theorem 2. If *j*-column of a matrix $A_{m\times n}$ dominates a convex linear combination of other columns of this matrix, and set $\{x^c, y^c, v_c\}$ is the decision of the game with a matrix *C*, which is received from a matrix by removal of *j*-column, is the arithmetic vector of dimension *n* by addition a coordinate equal to zero and located between coordinates and with numbers *j*-1 and j, then the set is the decision of the game with a matrix .

Theorem 3. If i-line of a matrix $A_{m\times n}$ is dominated by a convex linear combination of other lines of this matrix there will be such an optimum mixed strategy x^* of the first player in matrix game in which its x_i^* (i-coordinate) is equal to zero.

Theorem4. If in the game with matrix $A_{m\times n}$ its i-line is strictly dominated by a convex linear combination of other lines than i-coordinate is equal to zero in any optimum mixed strategy x^* of the first player.

Theorem 5. If in the game with matrix $A_{m\times n}$ its j-column strictly dominates a convex linear combination of other columns of this matrix than j-coordinate is equal to zero in any optimum mixed strategy y^* of the second player.

Important to notice that properties of solutions of the matrix game and the results received by the author of an article in theorems 1-5 help to find a solution of this matrix game by its successive simplification, i.e. by finding the solution of other game with a matrix of the smaller dimension. In essence, the process of simplification of the matrix game to another consists in work with a matrix of the given game. In particular, operation of an exception of pure players corresponds to the deletion of lines and columns from a matrix $A_{m\times n}$, thus it is necessary to consider that the exception of dominated strategy can lead to loss of some solutions of the given matrix game; however if only strictly dominated strategy are excluded than the set of solutions of the game does not change. After the operation of an exception of some lines and columns from the matrix $A_{m\times n} = (a_{ij})$ it is necessary to establish, whether the game with the received matrix H has the decision in pure strategy. For this purpose let's find whether the following parity is carried out or not

$$v_1 = \max_i \min_j h_{ij} = \min_j \max_i h_{ij} = v_2,$$

So we are checking, whether the bottom price of the game $v_1 = \max_i \min_j h_{ij}$ coincides with its top price $v_2 = \min_j \max_i h_{ij}$. If $v_1 = v_2$, game with matrix H has the solution in pure strategy, and therefore the given game with a matrix can be solved in pure strategy. If $v_1 \neq v_2$, it is necessary to resort to the mixed strategy, knowing that, according to Neumann's theorem, game in the mixed strategy has, at least, one decision. In that case, if the game is difficult, use methods of linear programming for a finding of the decision of the matrix game. However if the game is simple enough (but) and also has some foreseeable analytical structure, than sometimes it is possible to expect the optimum mixed strategy of players or a spectrum of optimum strategy. Of course last case is not the general. Providing that a spectrum of the mixed strategy of the first player in matrix game is understood as a set of all its pure strategy, and the probability of their application is positive, according to this strategy. Similarly we are defining a spectrum of the mixed strategy of the second player in the matrix game. We will notice that strictly dominated pure strategy of the player do not contain in a spectrum of its any optimum mixed strategy. These pure strategies are not applied, as their probabilities are equal to zero and are excluded in the process of simplification of the matrix game.

We will consider an example of finding solutions of the game which is settled by a matrix

$$A_{4\times 4} = \begin{pmatrix} 20 & 15 & 20 & 23 \\ 25 & \frac{35}{2} & 15 & 20 \\ 20 & 20 & 15 & 15 \\ 15 & 15 & 15 & \frac{25}{2} \end{pmatrix}.$$

In the given game $\Gamma_A = \{X, Y, H\}$ of the set $X = \{1, 2, 3, 4\}$, $Y = \{1, 2, 3, 4\}$ are sets of the pure strategy of players, and prize function is settled by a matrix $A_{4\times4}$.

It is possible to check up that the bottom price of the game $v_1 = 15$ is not equal to the top price of game $v_2 = 20$, therefore we will apply further mixed strategies. Game simplification is realizable by converting a matrix $A_{4\times4}$ as follows: at first to all elements of a matrix we will add figure $\alpha = -10$, and then the received matrix we will increase by number $k = \frac{1}{5} > 0$, as a result we will receive a matrix

$$B_{4\times4} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 3 & \frac{3}{2} & 1 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & \frac{1}{2} \end{pmatrix},$$

which sets the game. The optimum strategies of its players coincide to the optimum strategy of players in the game with a matrix $A_{4\times4}$. It is obvious that elements of the fourth line of a matrix $B_{4\times4}$ are not bigger than corresponding elements of the third line, therefore the third pure strategy of the first player dominates the fourth line. We will consider a convex linear combination of 1st, 2nd and 3rd lines - vectors of the matrix $B_{4\times4} : 0\times b_1 + 0\times b_2 + 1\times b_3$ with factors $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1$. It is obvious that this convex combination surpasses the fourth line of a matrix $B_{4\times4}$. Hence, game with a matrix C, which is received from a matrix $B_{4\times4}$ by the deletion of the fourth line, has the optimum mixed strategies x_c^*, y_c^*, which are optimum strategy and for the game with a matrix $B_{4\times4}$ if preliminary to add to the x^* vector the fourth zero co-ordinate.

Let's simplify now a matrix

$$C_{3\times4} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 3 & \frac{3}{2} & 1 & 2 \\ 2 & 2 & 1 & 1 \end{pmatrix}.$$

As elements of the first column of a matrix $C_{3\times4}$ are not less corresponding elements of the second column, than the second pure strategy of the second player dominates its first pure strategy. Let $1\times c_2 + 0\times c_3 + 0\times c_4$ is a convex linear combination of vectors-columns of the matrix $C_{3\times3}$ with factors $\beta_2 = 1$, $\beta_3 = 0$, $\beta_4 = 0$. It is visible that the first vector-column of matrix surpasses a convex combination of other columns. Having removed the first column, we will receive a matrix

$$D_{3\times3} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{2} & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

Optimum strategies x_D^*, y_D^* of the game with a matrix $D_{3\times3}$ are also optimum strategies for game with a matrix $C_{3\times4}$ if preliminary to add to the vector y_D^* the first zero coordinate. We will continue the analysis of vectors-lines of a matrix $D_{3\times3}$. It is obvious that a convex combination of the first and third lines with factors $\gamma_2 = \frac{1}{2}$, $\gamma_3 = \frac{1}{2}$: $\left(\frac{1}{2}d_1 + \frac{1}{2}d_3\right)$ surpasses the second vector-line. Having removed from the matrix $D_{3\times3}$ the second line, we will receive a matrix

$$E_{2\times3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix},$$

which sets the game, optimum strategy of which are also optimum strategy of the game with a matrix $D_{3\times3}$, if to consider the added second coordinate x_E^* of a vector equal to zero.

In a matrix $E_{2\times3}$ the third column surpasses a convex linear combination of other columns $0\times e_1 + 1\times e_2$ with factors *0, 1*. After removal of the third column we will receive a matrix

$$F_{2\times2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

which sets the game x_F^*, y_F^* optimum strategies of which will be optimum strategies also for the game with a matrix $E_{2\times3}$, if only to make one change: to consider the third coordinate y_F^* of a vector equal to zero.

So, the matrix $A_{4\times4}$ of the given game is transformed to a matrix $F_{2\times2}$ of other game which has no solutions in pure strategies. We will remind that the first line of a matrix $F_{2\times 2}$ corresponds to the 1st pure strategy of the first player, the second line of a matrix $F_{2\times 2}$ - 3rd pure strategy of the first player. Hence, the optimum mixed strategy of the first player of initial game possesses characteristic property: the second and fourth its co-ordinates are equal to zero:

$$x^* = (x_1^*, 0, x_3^*, 0).$$

Similarly first column of a matrix $F_{2\times 2}$ corresponds to the second pure strategy of the given game, the second column of a matrix corresponds to the third pure strategy of the given game, and the optimum mixed strategy of the second player of the given game looks like: , where the first and fourth coordinates are equal to zero. It is necessary to find the decision of game with a matrix . It is possible to prove that the set

$$\Gamma_F = \left\{x^* = \left(\frac{1}{2}, \frac{1}{2}\right), y^* = \left(\frac{1}{2}, \frac{1}{2}\right), v_* = \frac{3}{2}\right\}$$

is the solution of the given game as the matrix $F_{2\times 2}$ is symmetric. Thus an expected payoff of the first player in the situation (x^*, y^1) or in the situation (x^*, y^2), when the first player applies the mixed strategy $x^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, and the second player applies the first $y^1 = (1,0)$ or the second $y^2 = (0,1)$ pure strategy, will be calculated as follows:

$$F(x^*, y^*) = \sum_{j=1}^{2} a_{ij} x_j \times 1, \quad i=1,\ 2;$$

$$F(x^*, y^1) = a_{11}x_1 + a_{12}x_2 = 1\times\frac{1}{2} + 2\times\frac{1}{2} = \frac{3}{2};$$

$$F(x^*, y^2) = a_{21}x_1 + a_{22}x_2 = 2\times\frac{1}{2} + 1\times\frac{1}{2} = \frac{3}{2}.$$

In the same way can be proved that average of distribution of loss of the second player is equal in the game Γ_F to $\frac{3}{2}$, if its mixed strategy $y = (y_1, y_2)$ with $y_1 = \frac{1}{2}$, $y_2 = \frac{1}{2}$, and the first player applies any pure strategy.

At last, it is possible to assert that strategy $x^* = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ is an optimum strategy of the first player, and strategy $y^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ is an optimum strategy of the second player of this game with the matrix $A_{4\times 4}$. It is necessary to find the result of this game. It is obvious that the result of the game which is equal to $\frac{3}{2}$ and with the matrix $F_{2\times 2}$ coincides with the result of the game with a matrix $B_{4\times 4}$ because the matrix $A_{4\times 4}$ turns out from the matrix $B_{4\times 4}$ by multiplication of all its elements to figure k=5 and the subsequent addition to the received results figure б =10, i.e. any element a_{ij} of a matrix $A_{4\times 4}$ is equal to $a_{ij} = 5b_{ij} + 10$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$, where b_{ij} is the general element of a matrix $B_{4\times 4}$. Then the result of the game with a matrix $A_{4\times 4}$ is equal to

$$v_1^* = kv_* + \alpha = 5\times\frac{3}{2} + 10 = 17{,}5.$$

So, the solution of the given game with a matrix $A_{4\times 4}$ is $\{x^*, y^*, v_1^*\}$. Here is $x^* = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$, $y^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$, $v_1^* = 17{,}5$.

It is necessary to notice that while finding the solution of the given matrix game, solutions properties of the matrix game and the results of five theorems which are stated in this article by the author and which establish ties between decisions of two matrix games if lines or columns of these matrixes meet domination requirements are used.

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[1] See: *Neumann Dj., Morgenshtern O.* The theory of the games and economic behavior. M, 1970; *Djubin G. N., Suzdal V.G.* Introduction to the applied theory of the games. M, 1981; *Vorobev N.N.* Theory of the game for an economists and cybernetics. M, 1985; *Tchegodaev A.I.* Basis of the theory of the final antagonistic games and their application to the finding of solution problems of economy and military science: Study. Yaroslavl, 1993; *Kuzmina N.M.* System approach to the change's managing // Bulletin of the Samara State Economic University. Samara, 2007. џ 3 (29).